Recent Advancements to the Generalized High-Order Compact Scheme

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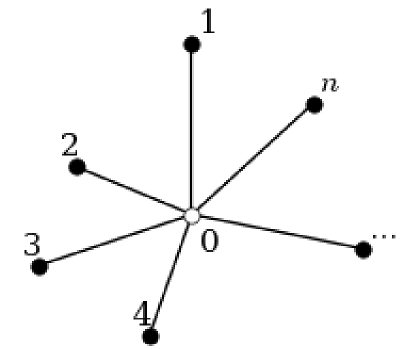
Background: GFDM

- **Generalized Finite Difference Method (GFDM)**
- **Tworzydlo, *IJNME*, 1988.**
- **Apply Taylor series approximations to unstructured grids**
  - At each grid point, form a matrix system:



$$\begin{bmatrix} g_{10}^{1} & g_{01}^{1} & g_{20}^{1} & g_{11}^{1} & g_{02}^{1} \\ g_{10}^{2} & g_{01}^{2} & g_{20}^{2} & g_{11}^{2} & g_{02}^{2} \\ g_{10}^{3} & g_{01}^{3} & g_{20}^{3} & g_{11}^{3} & g_{02}^{3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{10}^{n} & g_{01}^{n} & g_{20}^{n} & g_{11}^{n} & g_{02}^{n} \end{bmatrix}_0 \begin{bmatrix} \phi_x \\ \phi_y \\ \phi_{xx} \\ \phi_{xy} \\ \phi_{yy} \end{bmatrix}_0 \approx \begin{bmatrix} \phi_1 - \phi_0 \\ \phi_2 - \phi_0 \\ \phi_3 - \phi_0 \\ \vdots \\ \phi_n - \phi_0 \end{bmatrix}$$

$$g_{ij}^{m} = \frac{(x_m - x_0)^i}{i!} \frac{(y_m - y_0)^j}{j!}$$

$$\mathbf{G}_0 \delta\phi_0 \approx \Delta\phi_0$$

Background: GFDM

- **If this G_0 is invertible (n=5, restricted colinearity of grid points), then we can obtain stencil matrix $S_0 = G_0^{-1}$:**

$$\begin{bmatrix} \phi_x \\ \phi_y \\ \phi_{xx} \\ \phi_{xy} \\ \phi_{yy} \end{bmatrix}_0 \approx \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} \end{bmatrix}_0 \begin{bmatrix} \phi_1 - \phi_0 \\ \phi_2 - \phi_0 \\ \phi_3 - \phi_0 \\ \phi_4 - \phi_0 \\ \phi_5 - \phi_0 \end{bmatrix}$$

- **If n<5, then system is under-specified $\rightarrow$ invalid grid**

Background: GFDM

- If $n>5$, system is over-specified $\rightarrow$ one option is to use linear least-squares: $\mathrm{S}_0=(\mathrm{G}_0^{\mathrm{T}}\mathrm{G}_0)^{-1}\mathrm{G}_0$

- *Special case*: structured grid aligned with coordinate system $\rightarrow$ n=4, but G_0 is given by

$$\mathbf{G}_0 = \begin{bmatrix} (x_1 - x_0) & 0 & (x_1 - x_0)^2/2 & 0 & 0 \\ 0 & (y_2 - y_0) & 0 & 0 & (y_2 - y_0)^2/2 \\ (x_3 - x_0) & 0 & (x_3 - x_0)^2/2 & 0 & 0 \\ 0 & (y_4 - y_0) & 0 & 0 & (y_4 - y_0)^2/2 \end{bmatrix}$$

  – We can eliminate column 4 and ϕ_{xy} from $\delta\phi_0$ to obtain an invertible 4×4 system.

Recap of GFDM

- **For every grid point, we build a Taylor series matrix $\mathbf{G}_0$.**
- **For every $\mathbf{G}_0$, we obtain a stencil matrix S_0 (either $\mathbf{G}_0^{-1}$ or $(\mathbf{G}_0^{\mathrm{T}}\mathbf{G}_0)^{-1}\mathbf{G}_0$), which gives us a way to compute or express the derivatives of variables at each grid point.**
- **These expressions for derivatives can be used to build a global system to solve the governing PDE(s).**

Generalized High-Order Compact Scheme

- ***Question*: can we combine the ideas of high-order compact schemes and the GFDM?**
- ***Proposed approach*:**
  - **Extend $\delta\phi_0$ to include additional derivatives (typically through order 4).**
  - **Additional columns of G_0 can be obtained from Taylor series coefficients.**
  - **This forces the need for additional rows to obtain an invertible system.**
  - **Improve compactness by filling new rows of G_0 with derivatives of governing or auxiliary equations.**
  - ***Aside*: nonlinear problems would require solution of the G_0 *operators* at every iterate/time-step.**

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Simple 1D GHOC Example

- **1D convection-diffusion**

$$-\phi_{xx} + c\phi_x = f(x) \text{ on } x \in [0...1],$$
$$\text{with } \phi(0) = 0,\ \phi(1) = 1.$$

- **1D, nonuniform grid $\{x_i\}$, GHOC system:**

$$\begin{bmatrix} (x_{i-1}-x_i) & (x_{i-1}-x_i)^2/2 & (x_{i-1}-x_i)^3/6 & (x_{i-1}-x_i)^4/24 \\ (x_{i+1}-x_i) & (x_{i+1}-x_i)^2/2 & (x_{i+1}-x_i)^3/6 & (x_{i+1}-x_i)^4/24 \\ 0 & c & -1 & 0 \\ 0 & 0 & c & -1 \end{bmatrix}_i \begin{bmatrix} \phi_x \\ \phi_{xx} \\ \phi_{xxx} \\ \phi_{xxxx} \end{bmatrix}_i \approx \begin{bmatrix} \phi_{i-1}-\phi_i \\ \phi_{i+1}-\phi_i \\ f_x \\ f_{xx} \end{bmatrix}_i$$

- **Derivatives of *f(x)* can be obtained with GFDM.**

1D GHOC for General Linear Equation

- **The simplicity of the last two rows inspires more generalization:**

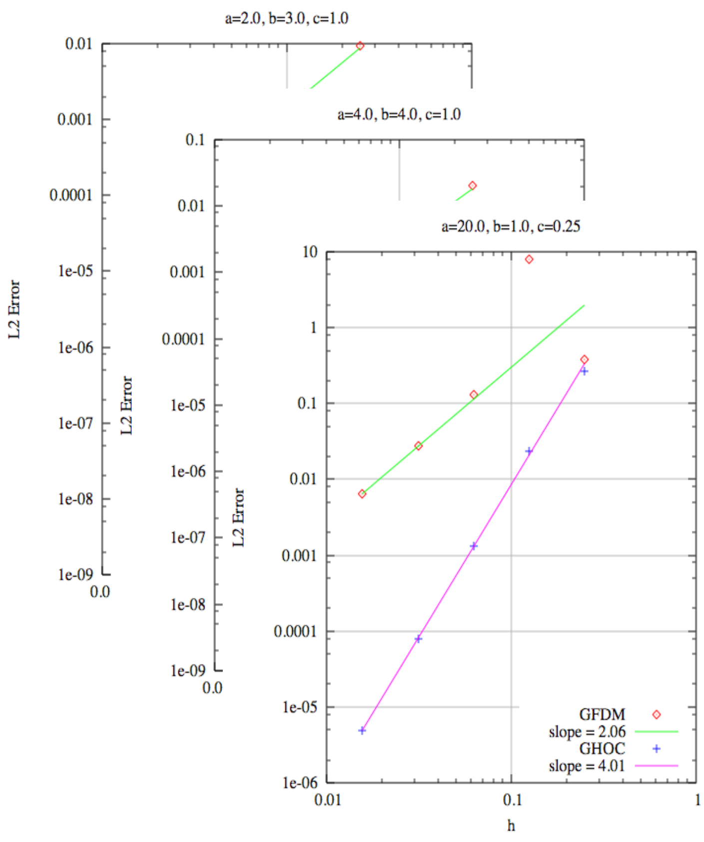
$$a(x)\phi + b(x)\phi_x + c(x)\phi_{xx} = f(x)$$

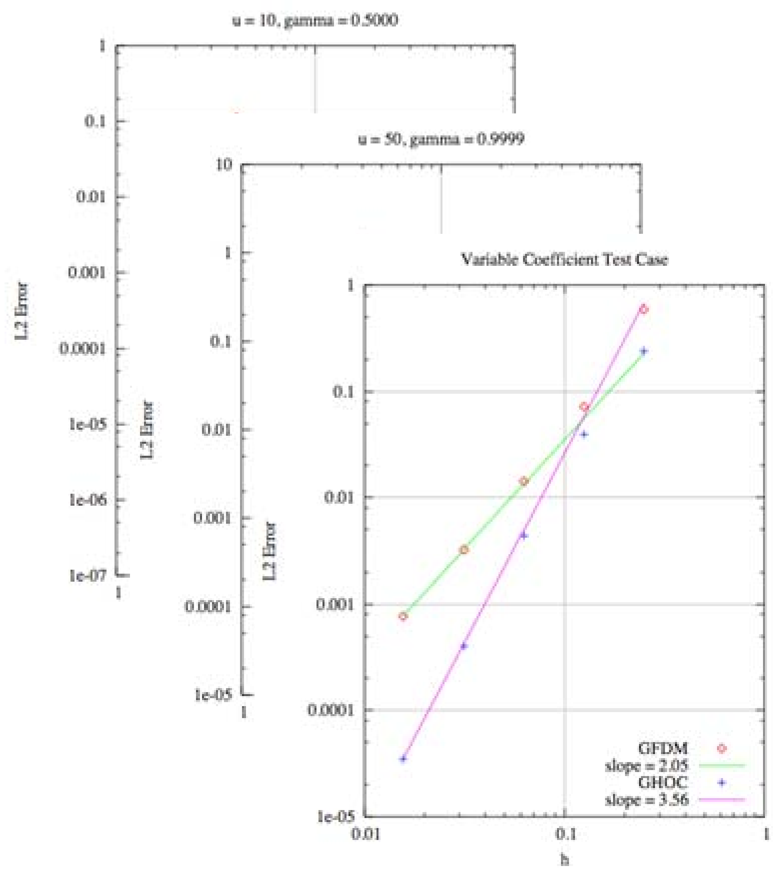
- **Yields GHOC system:**

$$\begin{bmatrix} (x_{i-1}-x_i) & (x_{i-1}-x_i)^2/2 & (x_{i-1}-x_i)^3/6 & (x_{i-1}-x_i)^4/24 \\ (x_{i+1}-x_i) & (x_{i+1}-x_i)^2/2 & (x_{i+1}-x_i)^3/6 & (x_{i+1}-x_i)^4/24 \\ a+b_x & b+c_x & c & 0 \\ 2a_x+b_{xx} & a+2b_x+c_{xx} & b+2c_x & c \end{bmatrix}_i \begin{bmatrix} \phi_x \\ \phi_{xx} \\ \phi_{xxx} \\ \phi_{xxxx} \end{bmatrix}_i = \begin{bmatrix} \phi_{i-1}-\phi_i \\ \phi_{i+1}-\phi_i \\ f_x - a_x\phi_i \\ f_{xx} - a_{xx}\phi_i \end{bmatrix}_i$$

1D GHOC Convergence Results

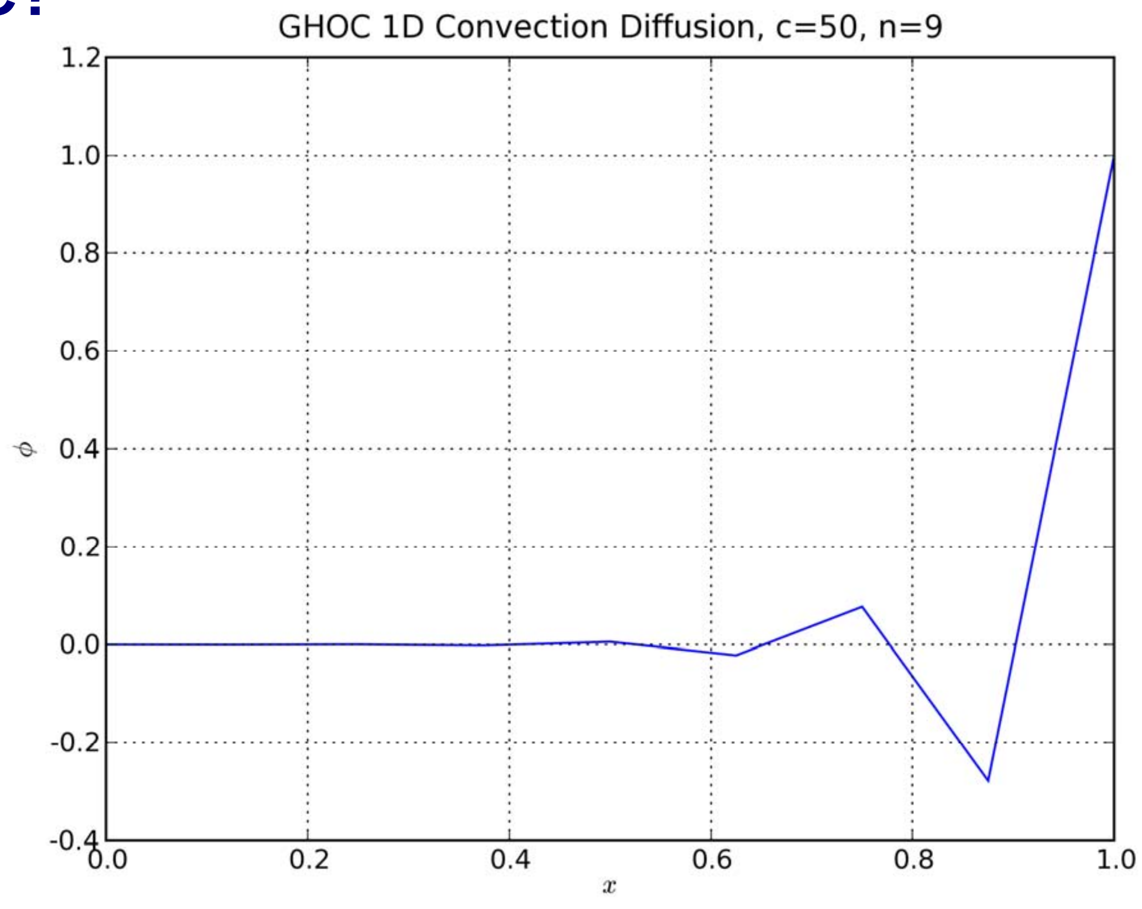
- **Previous results verify 4th-order accuracy:**





1D GHOC vs Classic HOC

- **Is GHOC on a uniform grid the same as classic HOC?**



Analysis of Simple 1D GHOC

- **If GHOC is not classic HOC, what is it?**
- **We can study GHOC system, restricted to simple convection-diffusion (constant c, f=0) on a uniform grid, to obtain the answer.**
- **Apply Gaussian elimination to obtain**

$$\begin{bmatrix} h & h^2/2 & h^3/6 & h^4/24 \\ 0 & h^2 & 0 & h^4/12 \\ 0 & 0 & h^2 & ch^4/12 \\ 0 & 0 & 0 & h^2(1+c^2h^2/12) \end{bmatrix} \begin{bmatrix} \phi_x \\ \phi_{xx} \\ \phi_{xxx} \\ \phi_{xxxx} \end{bmatrix}_i \approx \begin{bmatrix} \phi_{i-1} - \phi_i \\ \phi_{i+1} - 2\phi_i + \phi_{i-1} \\ c(\phi_{i+1} - 2\phi_i + \phi_{i-1}) \\ c^2(\phi_{i+1} - 2\phi_i + \phi_{i-1}) \end{bmatrix}$$

- **Solve for derivatives:**

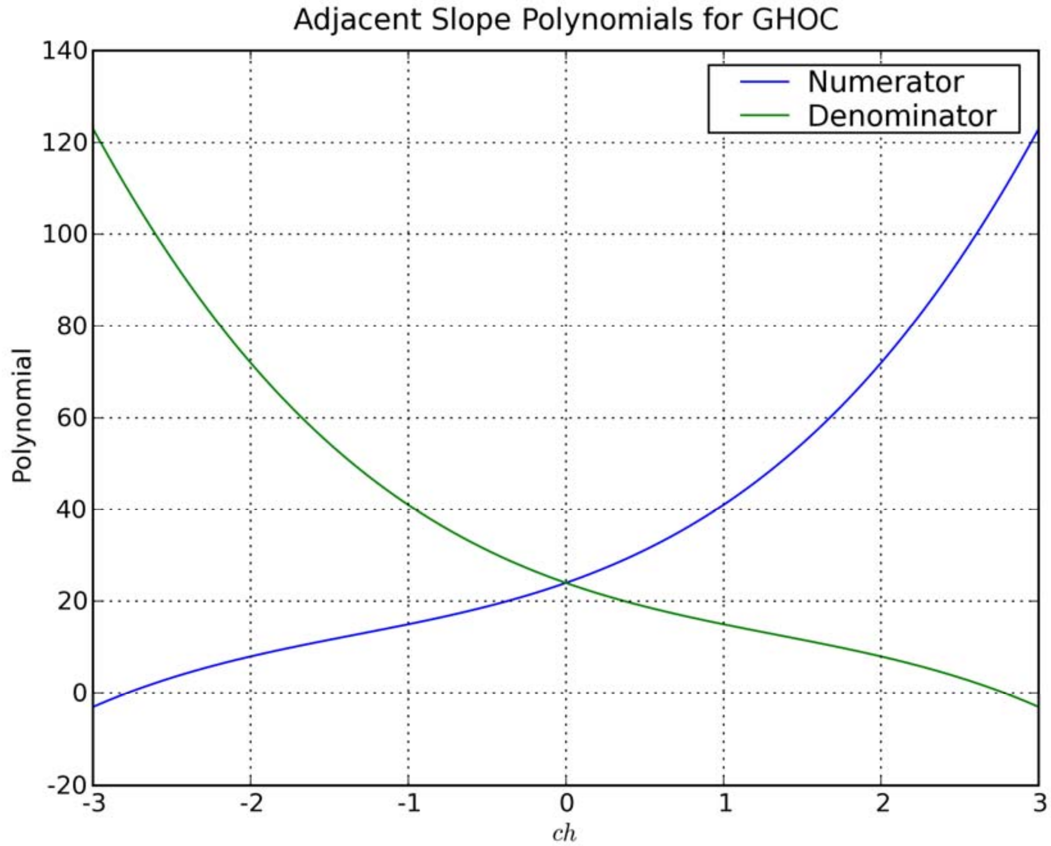
$$(\phi_x)_i \approx \delta_x \phi_i - \frac{2ch^2}{12 + c^2h^2} \delta_x^2 \phi_i,$$

$$(\phi_{xx})_i \approx \frac{12}{12 + c^2h^2} \delta_x^2 \phi_i.$$

Cell-Peclet Condition for 1D GHOC

- **Plug these into governing equation and solve for ratio of adjoining slopes:**

$$\frac{(\phi_{i+1}-\phi_i)/h}{(\phi_i-\phi_{i-1})/h}=\frac{c^3h^3+4c^2h^2+12ch+24}{-c^3h^3+4c^2h^2-12ch+24}$$



- **Roots (Cardano's method):**

$ch = \pm 2.9132412572848\ldots$

2D GHOC: Poisson's Equation

$$\nabla^2 \phi = f(x,y)$$

- **Extend $\delta\phi_0$: (2 first derivatives + 3 second derivatives + 4 third derivatives + 5 fourth derivates) = 14 elements in vector.**
- **Take derivatives of governing equation: (x, y, xx, xy, yy) = 5 rows.**
- **For an invertible system, we need (14-5) = 9 grid points in our stencil.**
- **Applying to a structured grid, we only have 8 neighboring grid points $\rightarrow$ have to play some kind of game to get a well-defined stencil.**

$$\begin{bmatrix} \phi_x \\ \phi_y \\ \phi_{xx} \\ \phi_{xy} \\ \phi_{yy} \\ \phi_{xxx} \\ \phi_{xxy} \\ \phi_{xyy} \\ \phi_{yyy} \\ \phi_{xxxx} \\ \phi_{xxxy} \\ \phi_{xxyy} \\ \phi_{xyyy} \\ \phi_{yyyy} \end{bmatrix}$$

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Conclusions

- Have extended GFDM to include HOC ideas
  - Transferred tedious part of derivation to the computer
- One dimension
  - Applied to completely general linear BVP
  - Nonuniform grids handled automatically
  - Fourth order convergence (demonstrated)
  - GHOC is *different* than classic HOC
- Two dimensions
  - Better understanding of matrix rank for standard, 9-point, orthogonal stencil
- Other extensions:
  - Time dependent: $f = S(x) - \partial\phi/\partial t$
  - Three dimensions
  - Nonlinear

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