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An Analysis of Stabilized Integration: Galerkin Meshfree Methods Discretizations for Advection Problems

Fifth WCCM / Meshless Methods

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Overview

- Background and motivation
- Formulation
  - Reproducing Kernel Particle Methods (RKPM)
  - Stabilized Conformal Nodal Integration (SCNI)
  - Von Neumann Analysis
- Results
  - Semi-Discrete dispersion characteristics.
  - Some transient behaviors.

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Why do this type of analysis?

- Provides a baseline for comparison to other methods.
- Aid in algorithmic design:
  - Mass matrix effects.
  - Requisite grid resolution.
  - Understand short wavelength behavior.
- Reliable tool, though other issues are also important for advective transport:
  - Monotonicity preserving.
  - Stability.

Reproducing Kernel Particle Method is used for the spatial discretization.

$$N_i^a(x_j) = \overline{\varphi}^a(x_j - x_i)\Delta_i = \overline{\varphi}_{i,j}^a \quad \Delta_i$$

$$\overline{\varphi}_{i,j}^a = [\beta^a]^t[\Theta^a] \qquad [\beta^a]^t = \{\beta_0, \beta_1\}$$

$$[\Theta^a] = \{\varphi_i^a, \varphi_i^a(x_j - x_i)\}$$

$$[m][\beta] = [P_o]$$

So that: $\mathbf{U}^a = \mathbf{N}^a\mathbf{d}^a$ Note: $U_j^a \neq d_j^a$

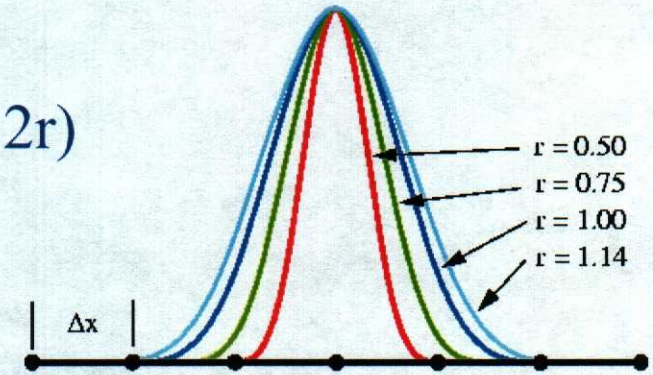
Bubnov-Galerkin: $\mathbf{M}\dot{\mathbf{d}} + \mathbf{A}\mathbf{d} = \mathbf{0}$

If domain is periodic and Δx, and r (a = 2r) constant

$$\mathbf{N}, \mathbf{A} \text{ and } \mathbf{M}$$

are Toeplitz and so commute. Thus,

$$\mathbf{M}\dot{\mathbf{U}} + \mathbf{A}\mathbf{U} = \mathbf{0}$$



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Stabilized Conforming Nodal Integration (Development for Diffusion Operator)

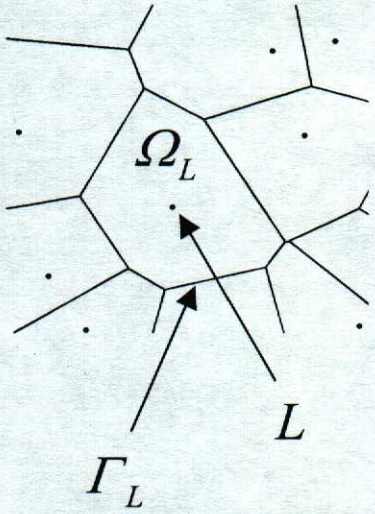
Integration constraint for diffusion operator (linear exactness):

$$\sum_{L=1}^{NIT} \nabla \Psi_i(x_L)\omega_L = 0 \quad \text{for} \quad \{i : supp(\Psi_i) \cap \Gamma = \varnothing\}$$

Assummed strain method satisfies constraint:

$$\tilde{\nabla} u^h(x_L) = \frac{1}{A_L} \int_{\Omega_L} \mathbf{n} u^h d\Gamma = \sum_i \tilde{\nabla} \Psi_i(x_L) d_i$$

$$\tilde{\nabla} \Psi_i(x_L) = \frac{1}{A_L} \int_{\Gamma_L} \Psi_i(x) \mathbf{n}(x) d\Gamma$$



The SCNI diffusion operator becomes:

$$K_{i,j} = \sum_{L=1}^{NP} \tilde{\nabla} \Psi_i(x_L) \tilde{\nabla} \Psi_j(x_L) A_L$$

The SCNI advection operator is approximated as:

$$A_{i,j} = \sum_{L=1}^{NP} \Psi_i(x_L) \tilde{\nabla} \Psi_j(x_L) A_L$$

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SCNI was developed for the diffusion operator, why use it for advection?

- Stabilized conformal nodal integration was developed to accurately integrate the diffusion operator. The linear exactness requirements of the diffusion operator are not needed for the advection equation for convergence.
- However, using the SCNI gradient operator for the advection problem may be advantageous for other reasons:
  - Avoid expensive direct evaluation of meshless shape function derivatives.
  - May already have the gradient operator 'B' if the problem incorporates physical or artificial (e.g. Petrov-Galerkin) diffusivity.
  - Will be cheaper than Gauss Quadrature.
  - May provide better accuracy than Direct Nodal Integration.

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The von Neumann analysis begins with spatial discretization.

- Select model equation:
  - first-order wave (scalar advection).

$$\frac{\partial U}{\partial t} + c_x \frac{\partial U}{\partial x} + c_y \frac{\partial U}{\partial y} = 0$$

- Spatial discretization:
  - Bubnov-Galerkin.
  - periodic (or infinite) domain.

$$\mathbf{M}\dot{\mathbf{U}} + \mathbf{A}(\mathrm{c})\mathbf{U} = \mathbf{0}$$

- Plane-wave solution:
  - substitute into semi-discrete form of PDE.
  - solve for dispersion.

$$U(x, y, t) = f(A, x, y, t, k, \omega)$$

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Phase speed is the projection of the fluid velocity in the wave direction.

- consider scalar advection:

$$\dot{T} + u_1 T_x + u_2 T_y = 0$$

- assume solution

$$T(\mathbf{x},t) = A\exp[\imath(k_1 x + k_2 y) - \imath\omega t]$$

- substitute and solve for ω:

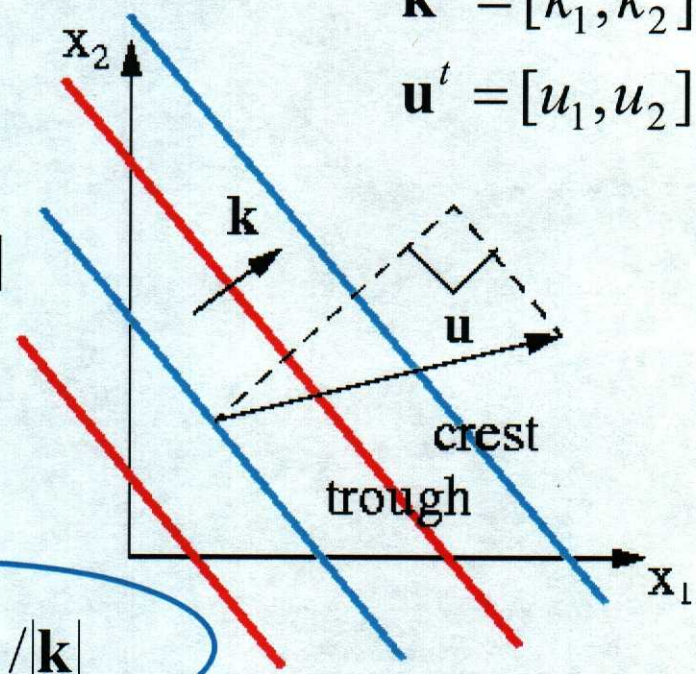
$$\omega = k_1 u_1 + k_2 u_2 = \mathbf{k}^t \mathbf{u}$$

$$T(x,t) = \exp[\imath \mathbf{k}^t (\mathbf{x} - \mathbf{u}t)]$$

$$c \equiv \mathbf{k}^t \mathbf{u} / |\mathbf{k}|$$

$$\mathbf{k}^t = [k_1, k_2]$$

$$\mathbf{u}^t = [u_1, u_2]$$



- For a non-dispersive fluid in a 1-D domain:

$$c = u_1 \qquad \text{so that} \qquad \psi \equiv u_1 / c = 1$$

(adapted from Gresho, Taiwan course, 1989).

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Group speed is the speed at which wavepacket energy is transported.

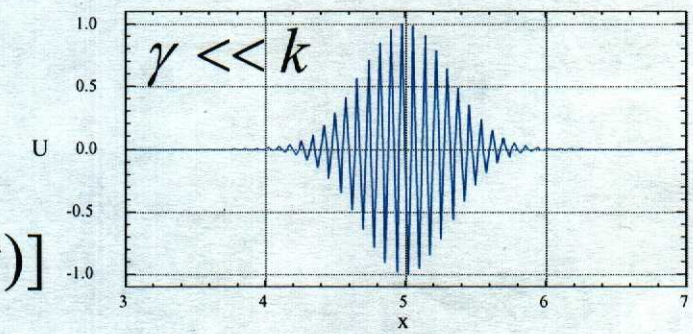
- group speed is defined as: $v_g \equiv \nabla_k[\mathbf{k}^t\mathbf{u}]$

  - in 1-D $\quad v_g = \partial[k\,\bar{c}(k)]/\partial k \qquad \zeta \equiv v_g / c$

- consider a short wavelength signal modulating a slowly varying envelope:

$$u(x,0) = \boxed{\sum_\gamma a_\gamma \exp[\imath\gamma x]}\exp[\imath k x]$$

$$u_\gamma(x,t) = a_\gamma \exp[\imath(\gamma + k)(x - \bar{c}(k+\gamma)t)]$$



$$u_\gamma(x,t) = \boxed{a_\gamma \exp[\imath\gamma(x - v_g(k)t)]}\ \boxed{\exp[\imath k(x - \bar{c}(k)t)]}$$

amplitude a_γ moving at v_g(k) $\qquad$ moving at phase speed

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Generate semi-discrete form of continuous plane wave solution.

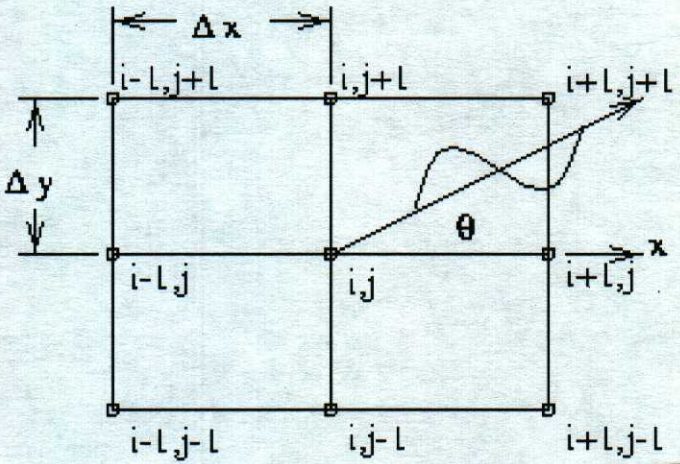
- **Continuous plane-wave solution:**

$$U(x,y,t) = \exp[\imath k(x\cos(\theta) + y\sin(\theta)) - \imath\omega t]$$

- **Discrete plane-wave solution:**

$$U_{(i+m,j+n)} = U_{(i,j)} \exp[\imath k(m\Delta x\cos(\theta) + n\Delta y\cos(\theta)) - \imath\overline{\omega} t]$$

$$\overline{\omega} = \overline{c}(k,\Delta x,\Delta y,\theta)\,k$$



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Assuming a 1-D domain (or θ = 0) the phase and group speeds become:

- **Normalized phase and group speeds are:**

$$\psi \equiv \frac{\bar{c}}{c} = \frac{f_a}{ckf_m} \qquad \zeta \equiv \frac{v_g}{c} = \frac{g_a f_m - f_a g_m}{c\, f_m^{\,2}}$$

where,

$$f_a = A_{i,i} + 2\sum_{l=1}^{n_x} \sin(kl\Delta x) A_{i,i+l}$$

$$f_m = M_{i,i} + 2\sum_{i=1}^{n_x} \cos(kl\Delta x) M_{i,i+l}$$

$$g_a \equiv \partial f_a / \partial k = 2\Delta x \sum_{l=1}^{n_x} l \cos(kl\Delta x) A_{i,i+l}$$

$$g_m \equiv \partial f_m / \partial k = -2\Delta x \sum_{l=1}^{n_x} l \sin(kl\Delta x) M_{i,i+l}$$

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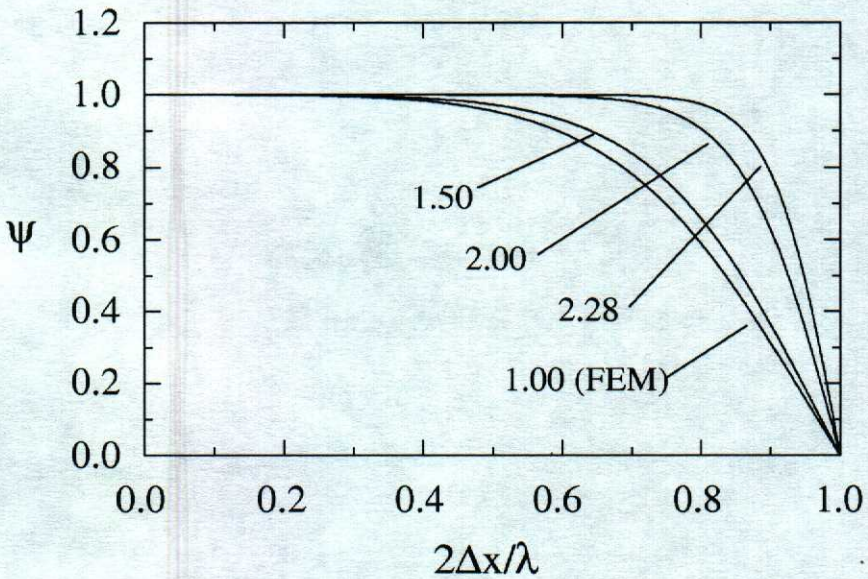
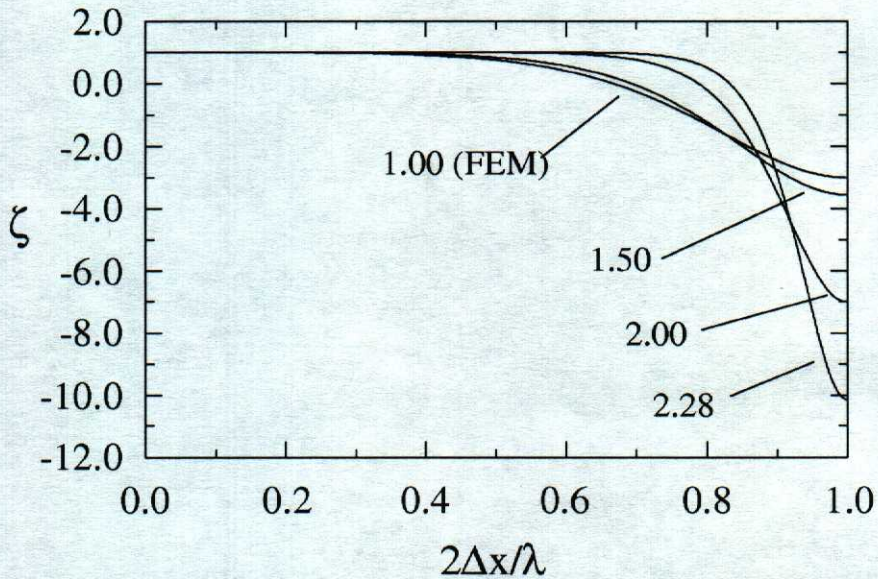
Studies demonstrate favorable dispersion properties for GI-RKPM relative to FEM.

Phase Speed



Group Speed



Full gauss quadrature for both mass and advection operators with consistent mass (CM) matrix.

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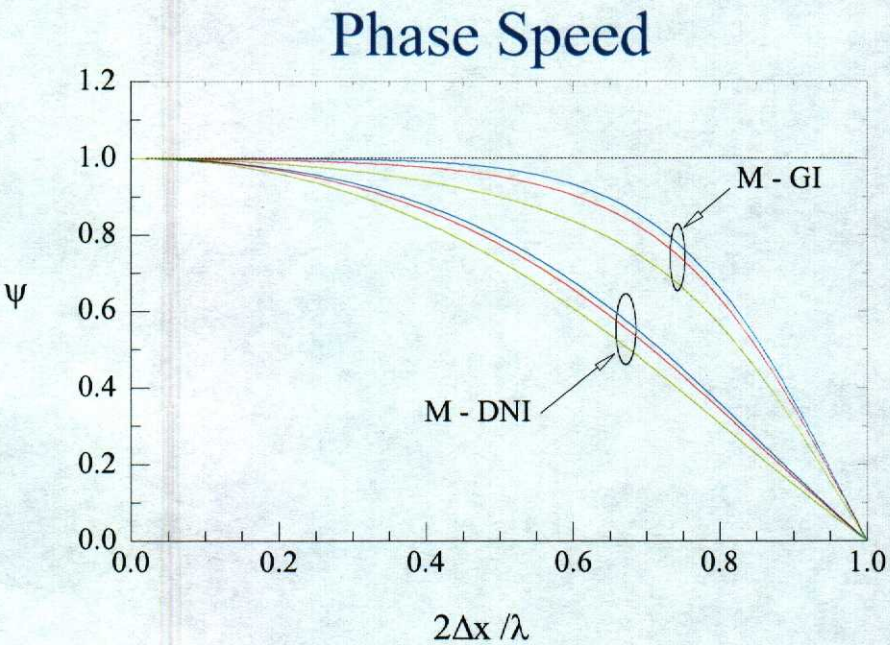
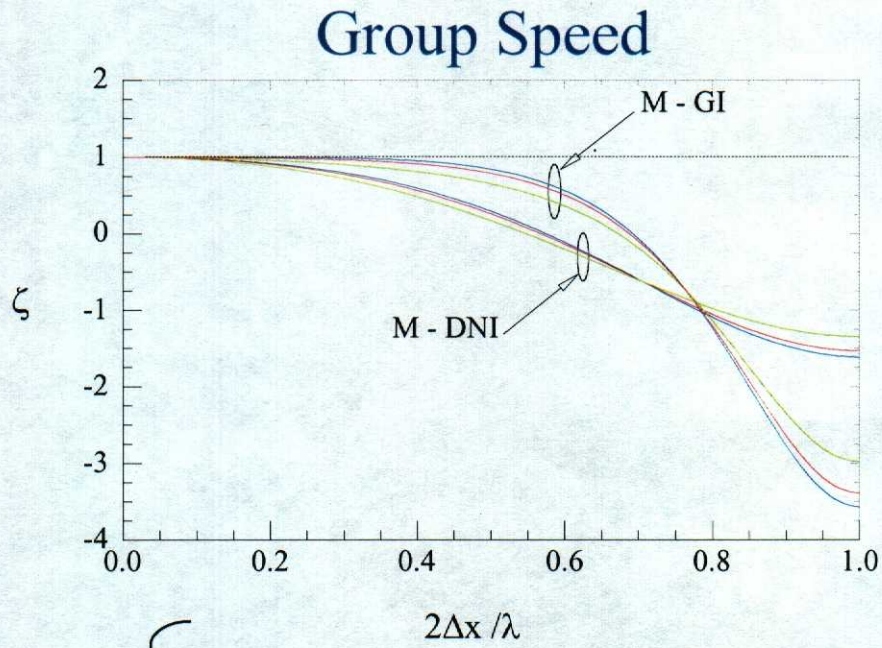
Dispersive character is strongly dependent on operator integration for consistent mass.

Results for consistent mass (CM), a = 1.50.





Advection operator integration scheme
- A-SCNI
- A-DNI
- A-GI

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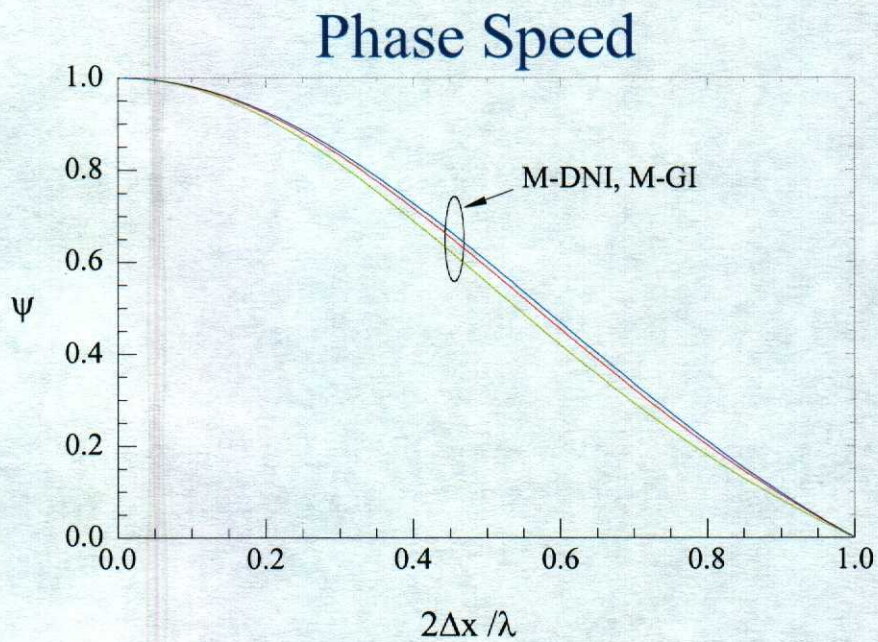
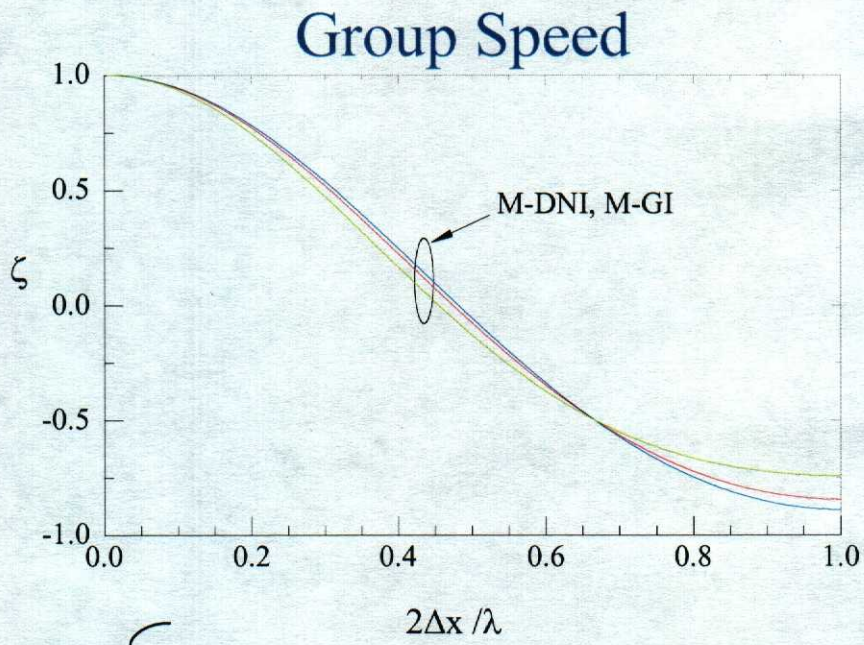
Lumped mass formulation indicates reduced dependence on integration technique.

Results for lumped mass (LM), a = 1.50.





Advection operator integration scheme
- A-SCNI
- A-DNI
- A-GI

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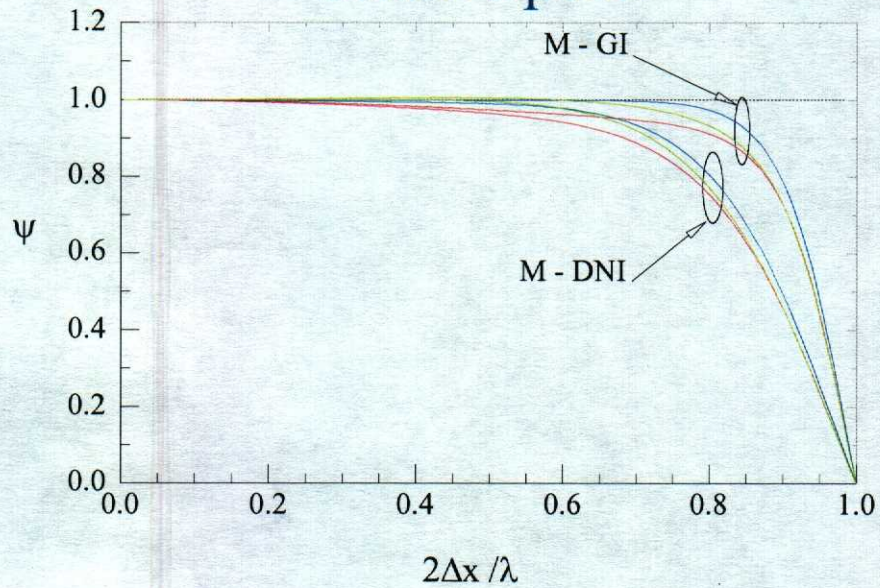
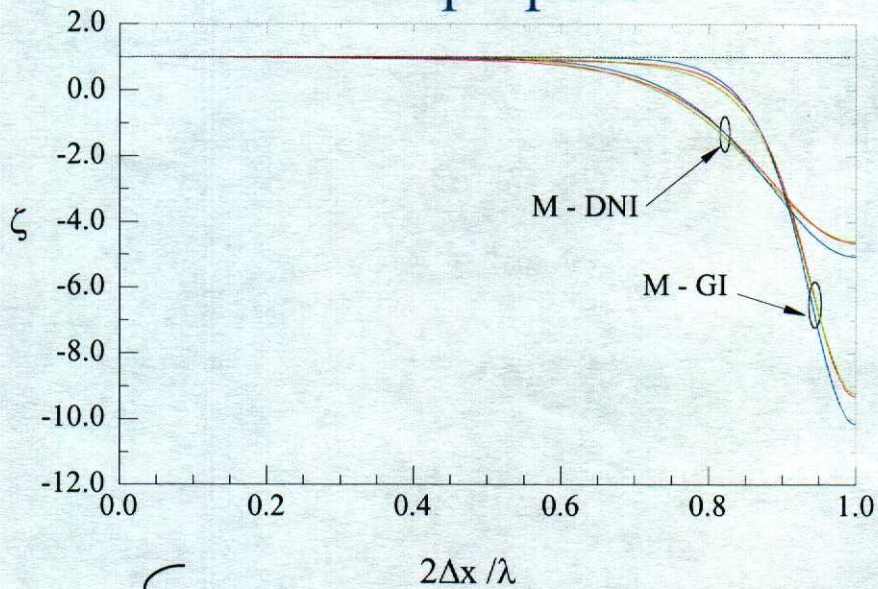
For CM, SCNI's phase degrades at larger kernel widths relative to A-GI and DNI.

Results for consistent mass (CM), a = 2.28.

Phase Speed



Group Speed



Advection operator integration scheme

- A-SCNI
- A-DNI
- A-GI

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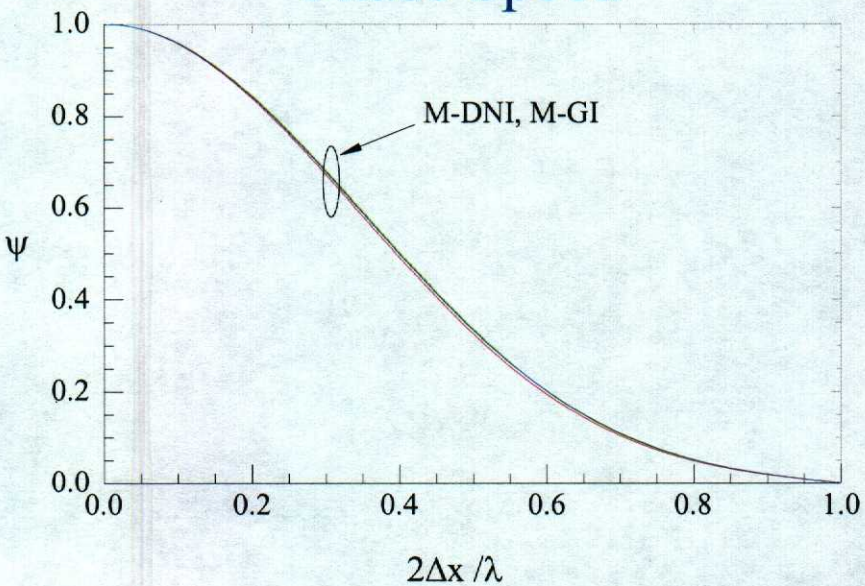
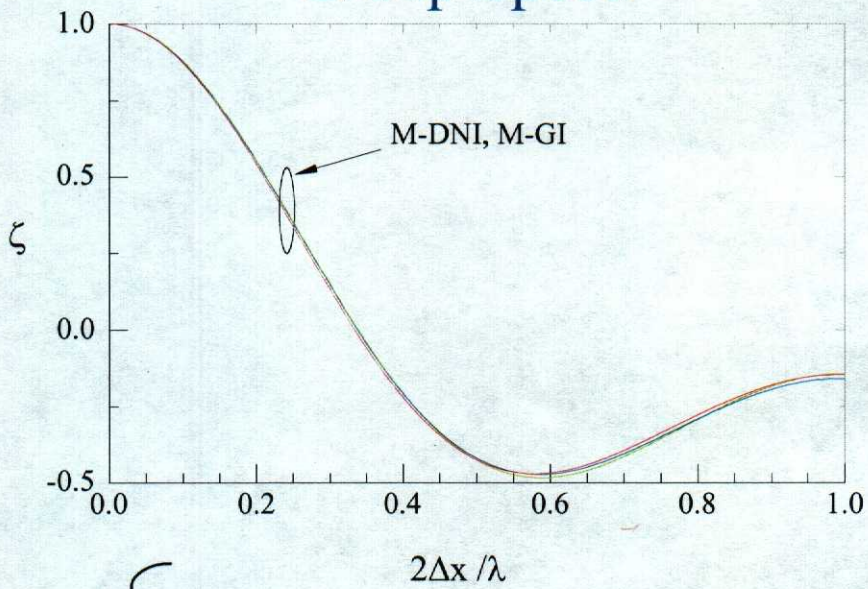
For LM, A-SCNI phase is identical to other techniques at large kernel width.

Results for consistent mass (LM), a = 2.28.

Phase Speed



Group Speed



Advection operator integration scheme

- A-SCNI
- A-DNI
- A-GI

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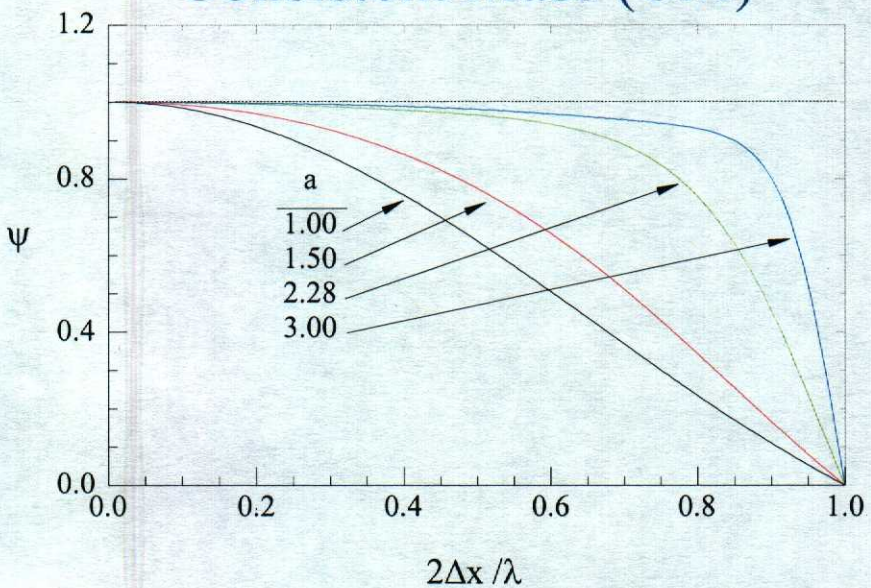
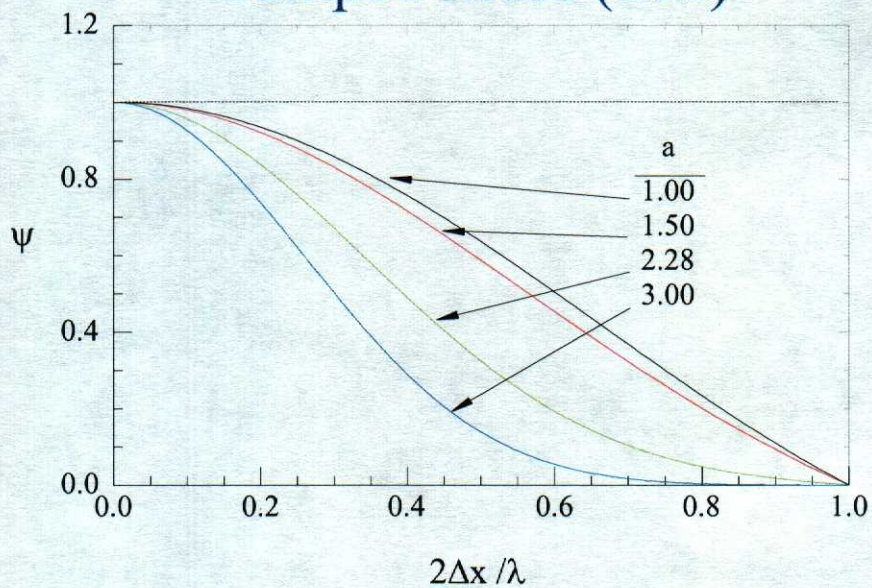
Kernel width impacts consistent and lumped mass phase behavior differently.

Phase speed results for M-DNI with A-SCNI.

Consistent Mass (CM)



Lumped Mass (LM)



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Uniform and Non-uniform transient solutions illustrate dispersion results.

1-D Advection Equation:

$$\dot{U} + cU' = 0 \qquad c = 1$$

Initial condition:

$$U(x,0) = sin(kx) \qquad k = 4\pi$$

Solutions:

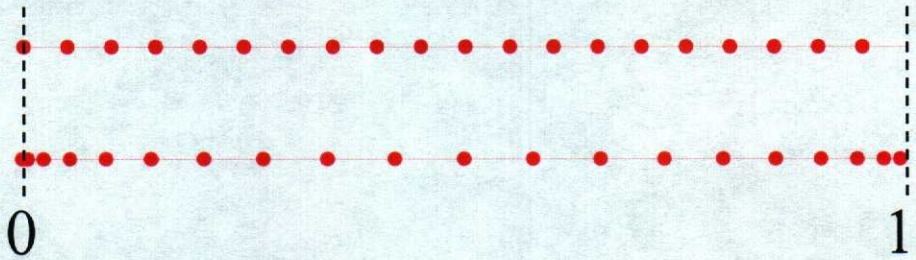
$$U_{exact}(x,t) = sin(k(x - ct))$$

$$U_{num}(x,t) = sin(k(x - \bar{c}_k t))$$

Periodic domain:

uniform grid

non-uniform grid



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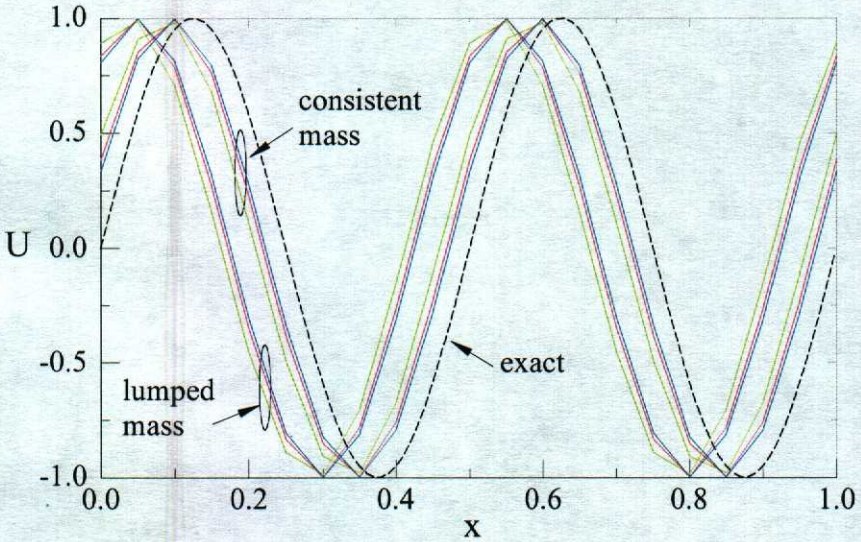
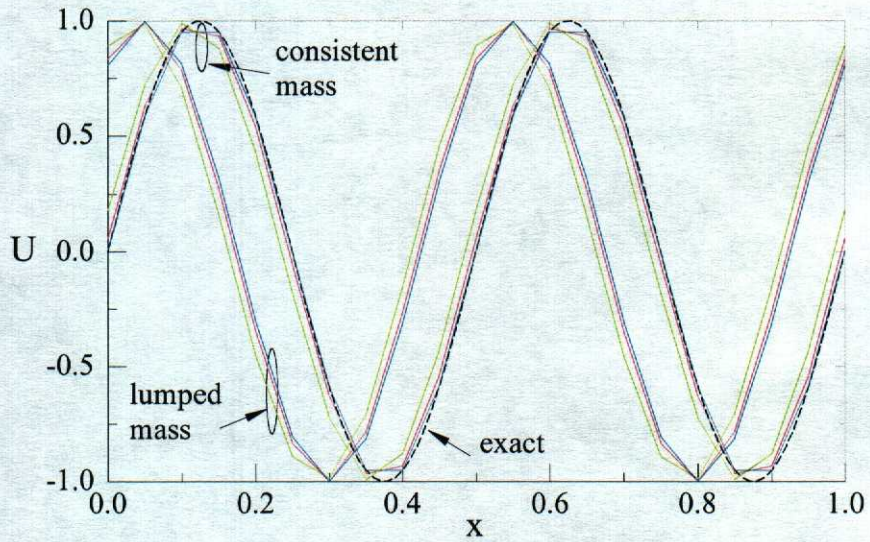
Single component, uniform mesh solution illustrates impact of integration and mass.

Results at t = 1 for discretizations with a = 1.5.

$$k\Delta x / \pi = 0.2$$



M-DNI Case



M-GI Case

Advection operator integration scheme
- A-SCNI
- A-DNI
- A-GI

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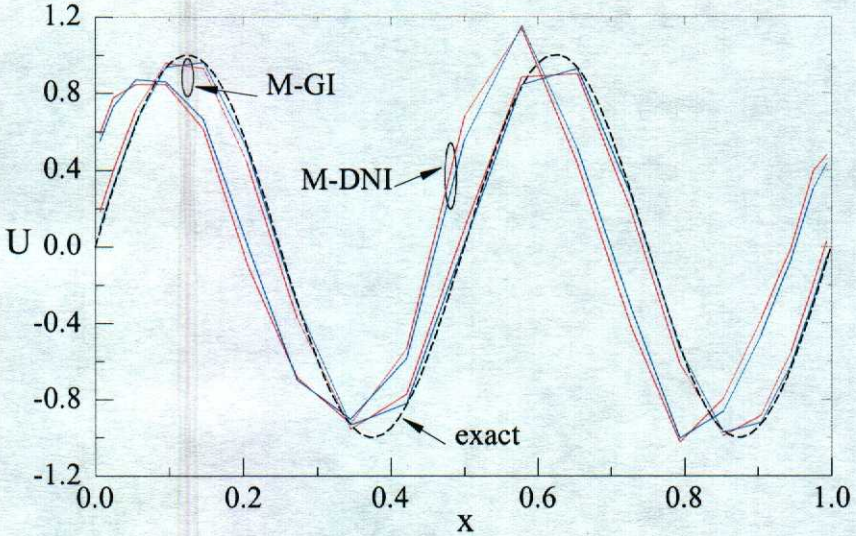
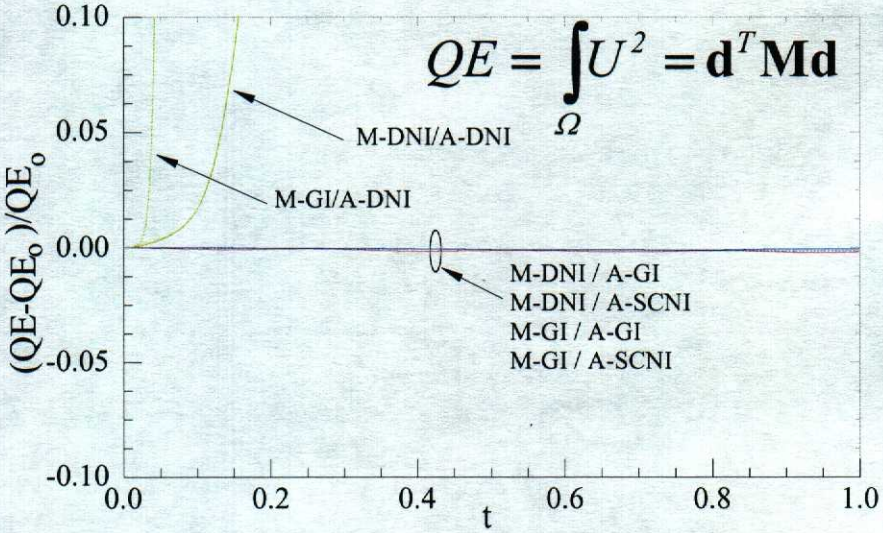
Single component, non-uniform mesh solution illustrates DNI issues for CM.

Consistent mass with a = 1.5.



Solution at t = 1



Energy change

Advection operator integration scheme
- A-SCNI
- A-DNI
- A-GI

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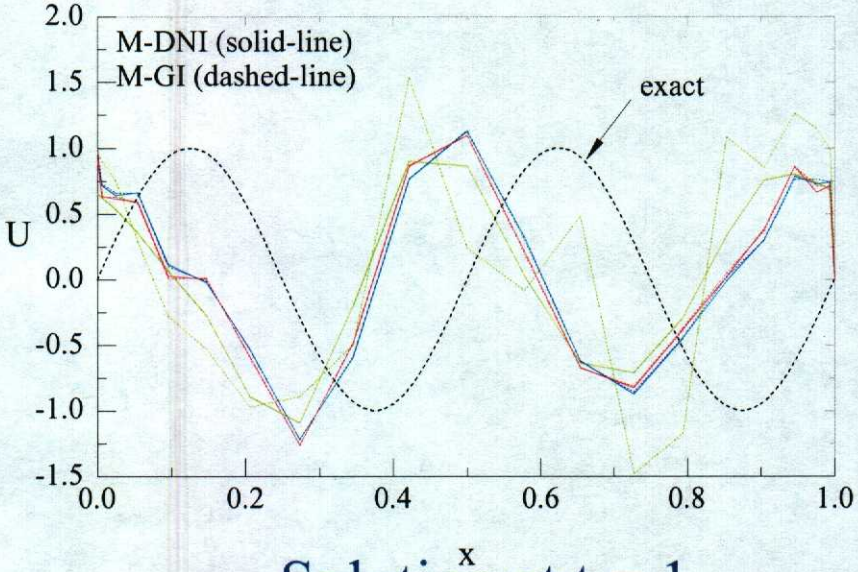
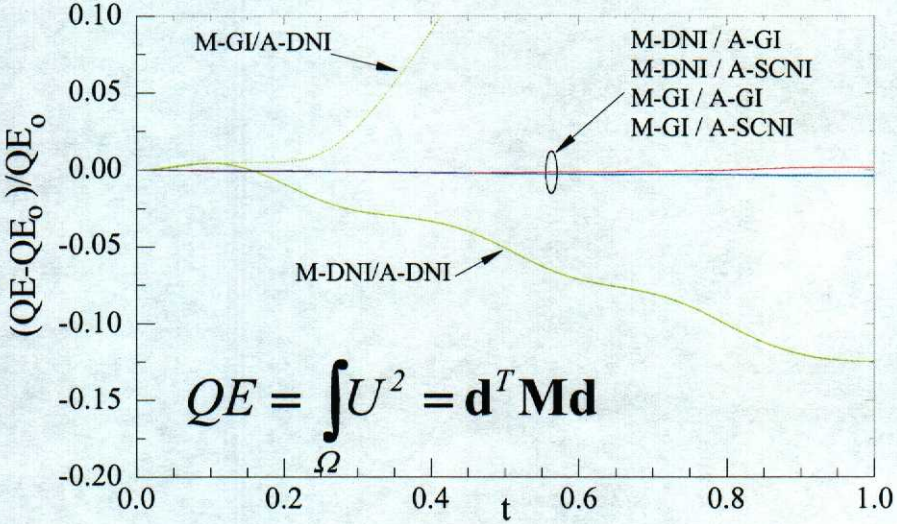
Single component, non-uniform mesh solution illustrates DNI issues for LM.

Lumped mass with a = 1.5.



Solution at t = 1



Energy change

Advection operator integration scheme

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- A-DNI
- A-GI

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Conclusions

- SCNI developed and produces excellent diffusive properties for diffusion PDE.
- Extending SCNI to advection PDE modifies dispersion properties relative to A-DNI and A-GI.
- A-SCNI produces dispersion properties similar in accuracy to those for 'fully' integrated gauss quadrature at significantly reduced cost.
- Lumping the mass, and/or using M-DNI has a much greater impact on dispersive properties than does using A-SCNI relative to A-GI.
- A-SCNI provides a stabilizing influence on our non-uniform grid results relative to A-DNI, most notably in conjunction with consistent mass.

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